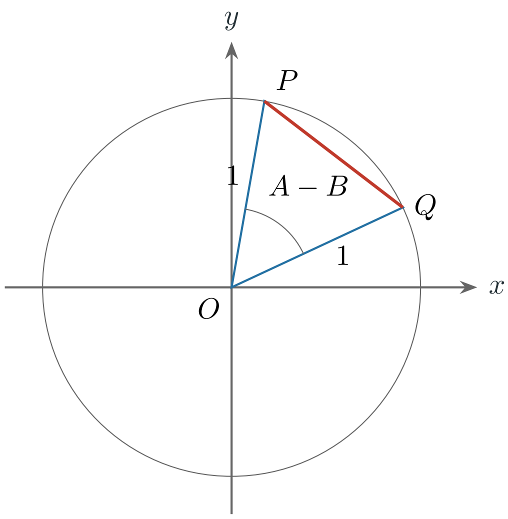
Part (c)

Game plan: Find the same chord length in two ways: using coordinates before and after a rotation. Equate the results.

Step 1: Choose two points on the unit circle

Let O be the origin and take $P = (\cos A, \sin A)$ and $Q = (\cos B, \sin B)$. Both points have distance 1 from O.



The diagram shows one convenient arrangement. The coordinates below are defined for general angles.

Step 2: Use the distance formula

$$\begin{aligned} |PQ|^2 &= (\cos A - \cos B)^2 + (\sin A - \sin B)^2 \\ &= \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B \\ &\quad - 2\cos A \cos B - 2\sin A \sin B \\ &= 2 - 2\cos A \cos B - 2\sin A \sin B. \end{aligned}$$

We used $\cos^2\theta + \sin^2\theta = 1$ for each angle.

Step 3: Find the same chord after rotating the circle

Rotate both points through $-B$. Rotation preserves distances. The image of Q is $(1, 0)$ and the image of P is $(\cos(A-B), \sin(A-B))$. Thus, equivalently to the cosine-rule calculation in the displayed triangle:

$$\begin{aligned} |PQ|^2 &= (\cos(A-B) - 1)^2 + \sin^2(A-B) \\ &= 2 - 2\cos(A-B). \end{aligned}$$

Step 4: Equate the two expressions

$$\begin{aligned} 2 - 2\cos(A-B) &= 2 - 2\cos A \cos B - 2\sin A \sin B, \\ \cos(A-B) &= \cos A \cos B + \sin A \sin B. \end{aligned}$$

This proves the required identity.